SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

BY

TOSIS KATO †

ABSTRACT

Recently B. Simon proved a remarkable theorem to the effect that the Schrödinger operator $T = -\Delta + q(x)$ is essentially selfadjoint on $C_0^\infty(R^m)$ if $0 \leqq q \in L^2(R^m)$. Here we extend the theorem to a more general case, $T = -\Sigma_{j=1}^m (\partial/\partial x_j - ib_j(x))^2 + q_1(x) + q_2(x)$, where b_j, q_1, q_2 are real-valued, $b_j \in C(R^m)$, $q_1 \in L^2_{loc}(R^m)$, $q_1(x) \geqq -q^*(|x|)$ with $q^*(r)$ monotone nondecreasing in r and $o(r^2)$ as $r \to \infty$, and q_2 satisfies a mild Stummel-type condition. The point is that the assumption on the local behavior of q_1 is the weakest possible. The proof, unlike Simon's original one, is of local nature and depends on a distributional inequality and elliptic estimates.

1. Introduction

Consider the Schrödinger operator on R^m of the form

$$T = -\sum_{k=1}^{m} (\partial_k - ib_k(x))^2 + q(x), \qquad \partial_k = \partial/\partial x_k, \tag{1.1}$$

where the b_k and q are real-valued functions such that

$$q \in L^2_{loc},\ b_k \in C^1, \qquad k = 1, \cdots, m; \tag{1.2}$$

here and in what follows C^1, L^2, L^2_{loc}, etc. refer to the whole space R^m unless otherwise indicated. Note that T is *formally self-adjoint.*

We denote by T_0 the operator T with $b_k = 0$, $k = 1, \cdots, m$.

We suppose that T is defined on the whole of L^2, with values in $\mathscr{D}'$ (the space of distributions on R^m). Indeed, $u \in L^1_{loc}$ already implies that $(\partial_k - ib_k)u$ is a distribution of order $\leqq 1$, and hence that $(\partial_k - ib_k)^2 u$ is a distribution of order $\leqq 2$ by $b_k \in C^1$. On the other hand, $u \in L^2$ implies $qu \in L^1_{loc}$ because $q \in L^2_{loc}$. Thus

† This work was partly supported by NSF Grant Gp-29369X.

we have $Tu \in \mathscr{D}'$. Obviously T is continuous on L^2 to $\mathscr{D}'$. (Actually $Tu \in \mathscr{D}'$ if only $u \in L^2_{loc}$, but we need not consider functions u outside L^2.)

Let $\dot{T}$ be the restriction of T with domain $\mathscr{D} = C_0^\infty$ (the space of test functions). Since $T\mathscr{D} \subset L^2$, $\dot{T}$ may be regarded as an operator in L^2. Then $\dot{T}$ is a symmetric operator. We are interested in the question whether or not $\dot{T}$ is *essentially self-adjoint.*

There are several well-known conditions equivalent to essential self-adjointness. Some of them involve $\dot{T}^*$, the adjoint (as an operator in L^2) of $\dot{T}$. In the present case, it is easily seen that $\dot{T}^*$ is exactly the restriction of T *in* L^2; in other words, $\dot{T}^*u$ is defined whenever u and Tu are in L^2, and in this case $\dot{T}^*u = Tu$.

Using this fact, we see immediately that the following conditions are equivalent.

(E1) $\dot{T}$ is essentially self-adjoint.

(E2) The restriction of T in L^2 is the (strong) closure of $\dot{T}$; in other words, for each $u \in L^2$ with $Tu \in L^2$, there exists a sequence $u_n \in \mathscr{D}$ such that $u_n \to u$ and $Tu_n \to Tu$ in L^2.

(E3) $(T + \zeta)\mathscr{D}$ is dense in L^2 for every complex number ζ with $\operatorname{Im} \zeta \neq 0$ or, equivalently, for two complex ζ with $\operatorname{Im} \zeta \gtrless 0$.

(E4) $T + \zeta$ is a one-to-one map of L^2 into $\mathscr{D}'$ for ζ as in (E3).

The problem of the essential self-adjointness of $\dot{T}$ has a long history, for which we refer the reader to a recent book by Schechter [1] and a recent paper by Simon [2]. But we have some comments. Condition (1.2) is not sufficient for $\dot{T}$ to be essentially self-adjoint, and many additional conditions have been proposed. But if $q = q^+ - q^-$ with $q^\pm \geqq 0$, then q^- required a stronger condition than q^+. For example, a *global* condition of Stummel type (see [1]) was required of q^- while only a *local* condition of the same type was required of q^+. In any case, some local condition stronger than $q^+ \in L^2_{loc}$ was assumed in all known theorems. (Incidentally, no restriction on the global behavior of the b_k was necessary, as was shown in Ikebe and Kato [3]; see also [1].)

Thus it came as a surprise when Simon [2] proved that $\dot{T}_0$ is essentially self-adjoint if $q = q_1 + q_2$ with $0 \leqq q_1 \in L^2$ and $q_2 \in L^\infty$. He further conjectured that $0 \leqq q_1 \in L^2_{loc}$ would suffice. His proof depends on a rather sophisticated tool developed in quantum field theory, which is functional analytic and therefore global in nature.

The purpose of the present paper is to show that $\dot{T}$ is, in fact, essentially self-adjoint under more general conditions than conjectured by Simon. Our assumptions are:

(C1) The b_k are real and in C^1.

(C2) $q = q_1 + q_2$, with $q_1 \in L^2_{loc}$ and

$$q_1(x) \geqq -q^*(|x|), \tag{1.3}$$

where $q^*(r)$ is monotone nondecreasing in $r > 0$ and $q^*(r) = o(r^2)$ as $r \to \infty$.

(C3) $q_2 \in L^2_{loc}$, with

$$\int_{|x| \leqq r} |q_2(x)|^2 dx \leqq K^2 r^{2s}, \qquad 1 \leqq r < \infty, \tag{1.4}$$

where K and s are some constants, and

$$\int_{|y| \leqq r} |q_2(x-y)| \, |y|^{2-m} dy \to 0 \text{ as } r \to 0, \text{ uniformly in } x \in R^m, \tag{1.5}$$

where $|y|^{2-m}$ should be replaced by $\log|y|$ if $m = 2$ and by 1 if $m = 1$.

If $m \geqq 5$, (C3) may be replaced by another condition:

(C3′) $q_2 \in L^{m/2}$.

Our theorem reads:

THEOREM. *$\dot{T}$ is essentially self-adjoint if* (C1), (C2), *and* (C3) *are satisfied. If $m \geqq 5$,* (C3) *may be replaced by* (C3′).

REMARKS. 1. The conditions (C1) to (C3) resemble those made in [3], but the local conditions on q are much weaker than in [3]. It may be noted that the Stummel-type condition (1.5) is usually introduced to define the Friedrichs extension of $\dot{T}$, but our aim here is more ambitious than that.

2. If $m = 4$, (C3′) does not work in our proof, since it corresponds to the critical case in the Sobolev inequality. It suffices to assume $q_2 \in L^p$ with some $p > 2$, but this implies (C3) so we need not consider it separately. Actually it suffices to assume that $q_2 \in L^p_{loc}$ *uniformly* (by which we mean that the L^p-norm of q_2 on any ball with radius 1 is uniformly bounded), since this implies (C3). Similarly, if $m \leqq 3$ it suffices that $q_2 \in L^2_{loc}$ uniformly, since this implies (C3). For these conditions, see [2].

3. One could consider a more general differential operator of the second order, as was done in [3], but we refrain from doing so in this paper.

The proof of the theorem given below is conventional and "local", based on standard elliptic estimates. But we use a lemma of a new type at a crucial stage. As in all problems of this kind, we have to prove, at some point, a result symbolically expressed by "weak = strong" (the theorem itself is of this nature). But the customary method does not seem to work owing to the high singularity of q_1.

Instead we use the following lemma which is stated in a form more general than necessary for the present purpose, since it has its own interest, it is useful for other applications, and since the proof is not essentially different from the special case

LEMMA A. *Let*

$$L = \sum_{j,k}^{1,m} (\partial_j - ib_j(x))a_{jk}(x)(\partial_k - ib_k(x)), \tag{1.6}$$

where a_{jk} and b_j are real-valued functions in $C^1(\Omega)$, Ω being an open set in R^m. Assume that $(a_{jk}(x))$ is a positive definite symmetric matrix for each $x \in \Omega$. If u and Lu are in $L^1_{loc}(\Omega)$, then

$$L_0|u| \geqq \mathrm{Re}\,[(\mathrm{sign}\,\bar{u})Lu], \tag{1.7}$$

where L_0 is the operator L with $b_k = 0$, $k = 1, \cdots, m$.

REMARKS. 1. By definition sign $\bar{u}(x) = \overline{u(x)}/|u(x)|$ if $u(x) \neq 0$ and $= 0$ if $u(x) = 0$.

2. Lu makes sense as a distribution on Ω if $u \in L^1_{loc}(\Omega)$, as was shown above. The assumption of the lemma implies that Lu is equal to a function in $L^1_{loc}(\Omega)$. Since sign $\bar{u}$ is bounded, the right hand side of (1.7) is in $L^1_{loc}(\Omega)$. The left hand side is in $\mathscr{D}'(\Omega)$ because $|u| \in L^1_{loc}(\Omega)$. The inequality (1.7) asserts that the difference of the two sides is a nonnegative distribution (hence a non-negative measure).

3. Technically, the lemma allows one to work in later proofs with inequalities that are almost linear in u, whereas most of the standard methods make use of inequalities that are quadratic in u, for example those involving Dirichlet integrals. But the singularity of q_1 does not allow one to define such quadratic expressions at crucial points.

The theorem will be proved in Sections 2 to 4 assuming Lemma A, which will be proved in Section 5.

Before closing this section, we give a simple formal identity which is used frequently in the sequel. For $u \in L^2$ and $\phi \in \mathscr{D}$, we have

$$T(\phi u) = \phi Tu - 2\sum_{k=1}^{m} (\partial_k\phi)(D_k u) - (\Delta\phi)u, \tag{1.8}$$

where $\Delta = \Sigma \partial_k^2$ is the Laplacian and

$$D_k = \partial_k - ib_k(x), \qquad k = 1, \cdots, m. \tag{1.9}$$

The proof of (1.8) is straightforward and may be omitted. The important fact is that on the right hand side the derivatives of u appear only in the combinations Tu and $D_k u$.

2. The special case $q^* = \text{const.}$

In this section we prove the theorem in the case $q^* = \text{const.}$ (so that q_1 is bounded from below) and deduce some estimates required later.

We introduce the new norm

$$\| u \|^2_{H^1_b} = \| u \|^2 + \sum_{k=1}^{m} \| D_k u \|^2, \qquad D_k = \partial_k - ib_k, \tag{2.1}$$

where $\| \ \|$ without subscript denotes the $L^2(R^m)$-norm. The completion of $\mathscr{D}$ under the norm (2.1) is denoted by H^1_b. It is easy to see that

$$H^1_b \subset H^1_{loc}, \qquad H^1_b \subset L^2, \tag{2.2}$$

algebraically and topologically, where H^1_{loc} is the usual Sobolev space.†

We note that H^1_b is, in general, not closed under complex conjugation.

PROPCSITION 1. *For $u \in H^1_b$ and $\varepsilon > 0$, we have*

$$\int |q_2(x)| \, |u(x)|^2 dx \leqq \varepsilon \| u \|^2_{H^1_b} + M_\varepsilon \| u \|^2, \tag{2.3}$$

where integration on R^m is understood and the constant M_ε depends only on ε and q_2.

PROOF. If $b = (b_k) = 0$, this is a well-known result. For the proof when (1.5) is assumed, see [1, Theorem 7.3]. When $m \geqq 5$ and (C3′) is assumed, it is a consequence of the Sobolev inequality. Indeed, $u \in H^1$ implies $u \in L^p$, $p^{-1} = 2^{-1} - m^{-1}$, with $\| u \|_{L^p} \leqq \text{const} \| u \|_{H^1}$, so that (2.3) holds at least with some $\varepsilon > 0$ depending on $\| q_2 \|_{L^{m/2}}$. But q_2 can be written as a sum $q' + q''$, where $q'' \in L^\infty$ and $\| q' \|_{L^{m/2}}$ is as small as one likes. This gives (2.3) for any $\varepsilon > 0$.

To prove (2.3) when $b \neq 0$, we may assume $u \in \mathscr{D}$. Then we apply (2.3) for $b = 0$ to $|u|$ replacing u. The left hand side is not changed by this substitution. Thus it suffices to show that $\| |u| \|_{H^1} \leqq \| u \|_{H^1}$. This is proved in [3] in a more general case.

PROPOSITION 2. *$\dot{T}$ is bounded from below.*

PROOF. Let $u \in \mathscr{D}$. Using (1.3) and (2.3), we have

$$(Tu, u) = \sum_{k=1}^{m} \| D_k u \|^2 + ((q_1 + q_2)u, u) \tag{2.4}$$

† As is easily seen, H^1_b is characterized as the set of all $u \in H^1_{loc}$ such that u and the $D_k u$ are in L^2.

$$\geqq \|u\|_{H_b^1}^2 - (q^* + 1)\|u\|^2 - \varepsilon\|u\|_{H_b^1}^2 - M_\varepsilon\|u\|^2$$
$$\geqq (1-\varepsilon)\|u\|_{H_b^1}^2 - (q^* + 1 + M_\varepsilon)\|u\|^2,$$

from which the result follows on choosing $\varepsilon = 1$.

PROPOSITION 3.† *For any complex number* ζ *with* $\mathrm{Re}\,\zeta$ *sufficiently large,* $T+\zeta$ *is a one-to-one map of* L^2 *into* $\mathscr{D}'$.

COROLLARY. $\dot{T}$ *is essentially* self-adjoint.

PROOF. (For the corollary, see conditions (E1) and (E4) of Introduction.) We have to show that $u \in L^2$ and $(T+\zeta)u = 0$ together imply $u = 0$.

$(T+\zeta)u = 0$ implies $Lu = (q+\zeta)u \in L^1_{loc}$, where L is the operator (1.6) with $a_{jk}(x) = \delta_{jk}$. Thus Lemma A gives

$$\Delta|u| \geqq \mathrm{Re}\,[(\mathrm{sign}\,\bar{u})(q+\zeta)u] = (q + \mathrm{Re}\,\zeta)\,|u| \geqq (-q^* + q_2 + \mathrm{Re}\,\zeta)|u|.$$

Writing $c^2 = \mathrm{Re}\,\zeta - q^*$, we have

$$(c^2 - \Delta)|u| \leqq -q_2|u| \leqq |q_2|\,|u|. \tag{2.5}$$

Suppose $\mathrm{Re}\,\zeta$ is so large that $c^2 > 0$. Then $c^2 - \Delta$ maps the Schwartz space $\mathscr{S}$ onto itself one to one, so that it also maps the dual space $\mathscr{S}'$ onto itself one to one. Furthermore, $(c^2-\Delta)^{-1}$ has a kernel $g_c(x-y)$, where

$$0 < g_c(x) \leqq A|x|^{2-m}e^{-c|x|/2} \qquad (m \geqq 3), \tag{2.6}$$

with A depending only on m. Inequality (2.6) follows from the expression $g_1(x) = \mathrm{const.}\,|x|^{(2-m)/2}K_{(m-2)/2}(|x|)$ (where K is the modified Bessel function of the second kind) and the homogeneity relation $g_c(x) = c^{m-2}g_1(cx)$. (It will not be necessary to describe how the following arguments should be modified in case $m \leqq 2$.)

It follows that $(c^2-\Delta)^{-1}$ is a positive operator, sending positive functions into positive functions and, consequently, positive elements of $\mathscr{S}'$ into positive ones. Thus (2.5) gives

$$|u| \leqq (c^2-\Delta)^{-1}|q_2 u|; \tag{2.7}$$

note that both $|u|$ and $|q_2u|$ are in $\mathscr{S}'$. This is trivial for $|u| \in L^2$. For $|q_2u|$, it follows from (1.4) if (C3) is assumed. Indeed (1.4) implies

$$\|q_2u\|_{L^1(B_R)} \leqq \|q_2\|_{L^2(B_R)}\|u\| \leqq KR^s\|u\|,$$

† This is our key proposition. Simon (private communication) has another proof when (C3′) (or its variant for $m \leqq 4$) is assumed, which also depends on Lemma A.

where B_R is the ball in R^m with center O and radius R. If (C3′) is assumed for $m \geqq 5$, we have similarly $\| q_2 u \|_{L^1(B_R)} \leqq \text{const. } R^{(m-4)/2} \| q_2 \|_L^{m/2} \| u \|$. Thus $q_2 u$ is a slowly increasing function in L^1_{loc} belonging to $\mathscr{S}'$.

In terms of the kernel g_c, (2.7) becomes

$$|u(x)| \leqq \int g_c(x-y) |q_2 u(y)| dy. \tag{2.8}$$

Set

$$v_c(x) = e^{-c|x|/4} |q_2 u(x)|. \tag{2.9}$$

Using the estimate on the local L^1-norm of $q_2 u$ given above, it is easy to see that $v_c \in L^1$. Multiplying (2.8) by $e^{-c|x|/4}|q_2(x)|$ and using the inequalities (2.6) and $e^{c(|y|-|x|)/4} \leqq e^{c|x-y|/4}$, we obtain

$$v_c(x) \leqq A\,|q_2(x)| \int |x-y|^{2-m} e^{-c|x-y|/4} v_c(y) dy,$$

where A is independent of c.

Integrating this inequality in x, we obtain

$$\| v_c \|_{L^1} \leqq A \int v_c(y) dy \int |x-y|^{2-m} e^{-c|x-y|/4} |q_2(x)| dx. \tag{2.10}$$

Now it is not difficult to show, using (1.5), that the integral in x on the right of (2.10) can be made smaller than $1/2A$ for every $y \in R^m$ if c is chosen large enough. (In this proof one should note that (1.5) implies that the integral of $|q_2|$ on any ball of a fixed radius is uniformly bounded.) Thus (2.10) gives $v_c = 0$. Then we see from (2.9) that $q_2 u = 0$, and from (2.8) that $u = 0$.

If (C3′) is assumed for $m \geqq 5$ instead of (C3), we argue as follows. $q_2 \in L^{m/2}$ as an operator in L^2 is bounded relative to $-\Delta$, with relative bound 0, by virtue of the Sobolev inequality (the proof parallels that of (2.3)). It follows that $Q_c = |q_2|(c^2 - \Delta)^{-1}$ is a bounded operator on L^2 to itself, with $\| Q_c \| < 1$ if c is sufficiently large. Since (2.7) implies $|u| \leqq Q_c^* |u|$ and hence $\| |u| \| \leqq \| Q_c^* |u| \|$ with $\| Q_c^* \| < 1$, we conclude that $|u| = 0$.

Since c can be made arbitrarily large by choosing $\operatorname{Re}\zeta$ large, this completes the proof of Proposition 3.

We need further estimates related to T for later use. To this end, it is convenient to introduce the Hilbert space H_b^{-1}, the anti-dual of H_b^1. Since $\mathscr{D} \subset H_b^1$ (algebraically and topologically), we have $H_b^{-1} \subset \mathscr{D}'$ and the elements of H_b^{-1} may be regarded as distributions on R^m. Since $H_b^1 \subset L^2$ densely, we have the standard triplet

$$H_b^1 \subset L^2 \subset H_b^{-1} \quad \text{(with dense embeddings)}. \tag{2.11}$$

PROPOSITION 4. *Let $u \in L^2$ and $Tu \in H_b^{-1}$. Then $u \in H_b^1$ with*

$$(2.12)\qquad (1-2\varepsilon)^{\frac{1}{2}}\|u\|_{H_b^1}\leqq(q^*+1+M_\varepsilon)^{\frac{1}{2}}\|u\|+(4\varepsilon)^{-\frac{1}{2}}\|Tu\|_{H_b^{-1}}.$$

PROOF. First assume $u\in\mathscr{D}$. From (2.4) we have

$$\operatorname{Re}((T+\zeta)u,u)\geqq(1-\varepsilon)\|u\|^2_{H_b^1}+(\operatorname{Re}\zeta-q^*-1-M_\varepsilon)\|u\|^2.$$

If $\operatorname{Re}\zeta$ is sufficiently large, we obtain

$$(1-\varepsilon)\|u\|^2_{H_b^1}\leqq\operatorname{Re}((T+\zeta)u,u)\leqq\|(T+\zeta)u\|_{H_b^{-1}}\|u\|_{H_b^1},$$

so that

$$(2.13)\qquad \|u\|_{H_b^1}\leqq(1-\varepsilon)^{-1}\|T+\zeta)u\|_{H_b^{-1}}.$$

If we assume $\operatorname{Re}\zeta$ to be larger yet if necessary, $(T+\zeta)\mathscr{D}$ is dense in L^2 (because $\dot{T}$ is essentially self-adjoint, see (E3) of Introduction) and hence in H_b^{-1} too. By the standard closure procedure using (2.13), we see that $T+\zeta$ maps a subset of H_b^1 onto H_b^{-1}. No other elements of L^2 are mapped into H_b^{-1}, since $T+\zeta$ is one to one on L^2.

Now let $u\in L^2$ with $Tu\in H_b^{-1}$. Then $(T+\zeta)u\in H_b^{-1}$, and it follows from the above result that $u\in H_b^1$.

To deduce the estimate (2.12), we may again assume $u\in\mathscr{D}$; in the general case we can argue by closure as above. We have by (2.4)

$$\begin{aligned}(1-\varepsilon)\|u\|^2_{H_b^1}&\leqq(q^*+1+M_\varepsilon)\|u\|^2+\|Tu\|_{H_b^{-1}}\|u\|_{H_b^1}\\&\leqq(q^*+1+M_\varepsilon)\|u\|^2+\varepsilon\|u\|^2_{H_b^1}+(4\varepsilon)^{-1}\|Tu^2\|_{H_b^{-1}},\end{aligned}$$

from which (2.12) follows.

3. Localization

In this section we continue to assume that $q^*=\text{const}$.

As above we denote by B_R the open ball in R^m with center O and radius R. If $0<r<R$, we denote by $\phi_{r,R}$ a function $\phi\in\mathscr{D}$ (not unique) such that $0\leqq\phi\leqq 1$, $\phi(x)=1$ for $x\in B_{r+\delta}$, and $\phi(x)=0$ for $x\notin B_{R-\delta}$, where $\delta=(R-r)/4$. It is easy to see that for $\phi_{r,R}=\phi$ we may assume that

$$(3.1)\qquad \|\phi\|_{L^\infty}=1,\quad \|\partial_k\phi\|_{L^\infty}\leqq c(R-r)^{-1},\quad \|\partial_j\partial_k\phi\|_{L^\infty}\leqq c(R-r)^{-2},$$

where c is a numerical constant.

PROPOSITION 5. *Let $u\in L^2$ and $Tu\in L^2_{loc}$. Then $u\in H^1_{loc}$, and we have for any $0<r<R$*

$$(3.2)\qquad \|u\|_{H_b^1(B_r)}\leqq 2^{\frac{1}{2}}(\|Tu\|_{L^2(B_R)}+q^{*\frac{1}{2}}\|u\|)+c\|u\|,$$

where c is a constant depending only on m, $R-r$, and q_2. Furthermore, $T(\phi u) \in L^2$ for any $\phi \in \mathscr{D}$.

PROOF. Let $\phi = \phi_{r,R}$ and apply (1.8). We have $\phi Tu \in L^2$ with $\| \phi Tu \| \leqq \| Tu \|_{L^2(B_R)}$, and $\| u\Delta\phi \| \leqq \| u \| \; \| \Delta\phi \|_{L^\infty}$. Furthermore, $\Sigma(\partial_k\phi)(D_k u)$ belongs to H_b^{-1}. To see this, let $\psi \in \mathscr{D}$ and estimate

$$\begin{aligned} |(\Sigma\,(\partial_k\phi)(D_k u), \psi)| &\leqq |\,\Sigma\,(u, D_k(\psi\partial_k\phi)| \\ &\leqq |\,\Sigma\,(u, (D_k\psi)(\partial_k\phi)) + (u, \psi\Delta\phi)| \\ &\leqq c' \| u \| \; \| \psi \|_{H_b^1}, \quad c' = (\Sigma \| \partial_k\phi \|_{L^\infty}^2 + \| \Delta\phi \|_{L^\infty}^2)^{\frac{1}{2}}. \end{aligned}$$

Hence $\Sigma\,(\partial_k\phi)(D_k u)$ belongs to H_b^{-1} with H_b^{-1}-norm $\leqq c' \| u \|$, and we see from (1.8) that $T(\phi u) \in H_b^{-1}$ with

$$\| T(\phi u) \|_{H_b^{-1}} \leqq \| Tu \|_{L^2(B_R)} + (2c' + c'') \| u \|, \qquad c'' = \| \Delta\phi \|_{L^\infty}.$$

It follows from Proposition 4 that $\phi u \in H_b^1$, with

$$(1-2\varepsilon)^{\frac{1}{2}} \| \phi u \|_{H_b^1} \leqq (q^* + 1 + M_\varepsilon)^{\frac{1}{2}} \| u \| + (4\varepsilon)^{-\frac{1}{2}}(\| Tu \|_{L^2(B_R)} + (2c' + c'') \; \| u \|).$$

Setting $\varepsilon = \frac{1}{4}$, we arrive at the desired estimate (3.2), in which $c = 2^{\frac{1}{2}}[(2c' + c'') + (1 + M_{\frac{1}{4}})^{\frac{1}{2}}]$. Here c' and c'' depend only on m and $R - r$, and $M_{\frac{1}{4}}$ depends only on q_2.

Once we know that $u \in H_{loc}^1$, the right hand side of (1.8) belongs to L^2 for any $\phi \in \mathscr{D}$. This completes the proof of Proposition 5.

PROPOSITION 6. *Let $u \in L^2$, $Tu \in L^2$, and* supp $u \subset B_R$. *Then there exists a sequence $u_n \in \mathscr{D}$ such that* supp $u_n \subset B_R$, *$u_n \to u$ and $Tu_n \to Tu$ in L^2.*

PROOF. If we disregard the condition supp $u_n \subset B_R$, there exists such a sequence because $\dot{T}$ is essentially self-adjoint (see (E2) of Introduction). Suppose such a sequence $\{u_n\}$ has been constructed.

Let $\phi \in \mathscr{D}$ be such that supp $\phi \subset B_R$ and $\phi = 1$ on supp u. Then $\phi u_n \to \phi u = u$ in L^2. Next replace u by u_n in (1.8) and let $n \to \infty$. Then $\phi T u_n \to \phi T u = Tu$ in L^2; note that supp $Tu \subset$ supp u. Also $u_n \Delta\phi \to u\Delta\phi = 0$, since $\Delta\phi = 0$ on supp u. Thus $\{\phi u_n\}$ will satisfy all the requirements for $\{u_n\}$ of the proposition if we show that

$$(\partial_k\phi)(D_k u_n) \to 0 \text{ in } L^2. \tag{3.3}$$

Applying (2.12) to $u_n - u$, we see that $u_n \to u$ in H_b^1. Hence $D_k u_n \to D_k u$ in L^2, so that $(\partial_k\phi)(D_k u_n) \to (\partial_k\phi)(D_k u) = 0$ in L^2; note that $\partial_k\phi = 0$ on supp $u \supset$ supp $D_k u$. This proves (3.3).

4. The general case

We are now in a position to prove the theorem in the general case.

PROPOSITION 7. *Proposition 5 is true in the general case with the single change: replace q^* by $q^*(R)$ in* (3.2).

PROOF. We define a new operator T' of the same form as T by changing q_1 into q_1', where

$$q_1'(x) = \begin{cases} q_1(x), & x \in B_R, \\ q_1(x) + q^*(|x|) - q^*(R), & x \notin B_R. \end{cases}$$

Then $q_1'(x) \geqq -q^*(R)$ everywhere, so that the results of the preceding sections apply to T' with $q^* = q^*(R)$.

Let $u \in L^2$ and $Tu \in L^2_{loc}$. Since $q_1' - q_1$ is a locally bounded function, $T'u = Tu + (q_1' - q_1)u$ is in L^2_{loc}. It follows from Proposition 5 applied to T' that $u \in H^1_{loc}$ and that (3.2) holds with T and q^* replaced by T' and $q^*(R)$, respectively. But since $T'u = Tu$ in B_R, we have $\| T'u \|_{L^2(B_R)} = \| Tu \|_{L^2(B_R)}$. This proves (3.2) with q^* replaced by $q^*(R)$. The last assertion of Proposition 5 can be proved as before.

PROPOSITION 8. *Let $u \in L^2$, $Tu \in L^2$, and $\phi \in \mathcal{D}$. Then $T(\phi u) \in L^2$, and there is a sequence $u_n \in \mathcal{D}$ such that $u_n \to \phi u$ and $Tu_n \to T(\phi u)$ in L^2.*

PROOF. Choose $R > 0$ such that $\operatorname{supp} \phi \subset B_R$. Let T' be the operator introduced in the proof of Proposition 7. Since $\operatorname{supp} \phi u \subset B_R$, Proposition 6 shows that there is a sequence $u_n \in \mathcal{D}$ such that $\operatorname{supp} u_n \subset B_R$, $u_n \to \phi u$ and $T'u_n \to T'(\phi u)$ in L^2. Then $Tu_n = T'u_n \to T''(\phi u) = T(\phi u)$, as required.

PROPOSITION 9. *There is a sequence $\phi_n \in \mathcal{D}$ with the following properties. If $u \in L^2$ and $Tu \in L^2$, then $\phi_n u \to u$ and $T(\phi_n u) \to Tu$ in L^2.*

PROOF. Let $\phi = \phi_{1,2}$ as defined in the beginning of Section 3, and set $\phi_n(x) = \phi(x/n)$. Then $\phi_n \to 1$, $\partial_k \phi_n \to 0$, and $\partial_j \partial_k \phi \to 0$ boundedly. Hence $\phi_n u \to u$ in L^2.

To prove that $T(\phi_n u) \to Tu$, we apply (1.8) with ϕ replaced by ϕ_n and let $n \to \infty$. Then $\phi_n Tu \to Tu$ and $u \Delta \phi_n \to 0$ by the dominated convergence theorem. Thus it suffices to show that the remaining terms $(\partial_k \phi_n)(D_k u)$ also tend to 0 in L^2.

We have $\|(\partial_k \phi_n)(D_k u)\| \leqq \| \partial_k \phi_n \|_{L^\infty} \| D_k u \|_{L^2(B_{2n})}$ because $\operatorname{supp} \phi_n \subset B_{2n}$. But $\| \partial_k \phi_n \|_{L^\infty} \leqq \text{const.}/n$, as is easily seen from the definition. On the other hand, application of Proposition 7 with $r = 2n$ and $R = 2n + 1$ gives

$$\| D_k u \|_{L^2(B_{2n})} \leqq \| u \|_{H^1_0(B_{2n})} \leqq 2^{\frac{1}{2}}(\| Tu \| + q^*(2n+1)^{\frac{1}{2}} \| u \|) + c \| u \|,$$

where c does not depend on n because $R - r = 1$. Since $q^*(2n+1)^{\frac{1}{2}} = o(n)$ by hypothesis, the desired result follows.

PROPOSITION 10. *Let $u \in L^2$ and $Tu \in L^2$. Then there is a sequence $u_n \in \mathscr{D}$ such that $u_n \to u$ and $Tu_n \to Tu$ in L^2.*

COROLLARY. *$\dot{T}$ is essentially self-adjoint. (This completes the proof of the theorem; see condition (E2) in Introduction.)*

PROOF. According to Proposition 9, for each $n = 1, 2, \cdots$ there is $\phi_n \in \mathscr{D}$ such that $\| \phi_n u - u \| < 1/n$ and $\| T(\phi_n u) - Tu \| < 1/n$. For this ϕ_n, Proposition 8 shows that there is $u_n \in \mathscr{D}$ such that $\| u_n - \phi_n u \| < 1/n$ and $\| Tu_n - T(\phi_n u) \| < 1/n$. Hence $\| u_n - u \| < 2/n$ and $\| Tu_n - Tu \| < 2/n$. This proves the proposition and hence the theorem.

5. Proof of Lemma A

We prepare several lemmas before proceeding to the proof of Lemma A. In what follows L^1_{loc} means $L^1_{loc}(\Omega)$.

LEMMA 1. *Let u and Lu be in L^1_{loc}. Then $\partial_k u \in L^1_{loc}$, $k = 1, \cdots, m$.*

PROOF. This is a well-known result for elliptic differential operators, at least if the a_{jk} and b_j are smooth enough. Since our assumptions on them are rather weak, we shall briefly indicate how it can be proved. Construct a parametrix $g(x,y)$ as the standard fundamental solution for L_0 with the coefficients frozen at x. Then $Lg(x,\cdot) = -\delta_x + k(x,\cdot)$, where $k(x,y)$ is C^1 in x for $x \neq y$ and has a singularity $O(|x-y|^{1-m})$. The standard procedure gives a local integral representation for u in terms of Lu and u itself. After an iteration of this formula, the kernels are at most $O(|x-y|^{2-m})$ (assuming $m \geqq 3$), and one can compute $\partial_k u$ by differentiating under the integral sign. This leads to the desired result. (Actually it follows that $u \in L^p_{loc}$ and $\partial_k u \in L^s_{loc}$ with any $p^{-1} > 1 - 2m^{-1}$, $s^{-1} > 1 - m^{-1}$.)

The next lemma concerns the well-known Friedrichs mollifier J^ρ, which is an integral operator with a nonnegative kernel $j^\rho(x-y)$, with $j^\rho \in \mathscr{D}$ supported on B_ρ and with L^1-norm equal to 1 (see Friedrichs [4]). We write $J^\rho w = w^\rho$. Note that $w^\rho \in C^\infty(\Omega_\rho)$ for each $w \in \mathscr{D}'(\Omega)$, where Ω_ρ is a domain slightly smaller than Ω and $\Omega_\rho \to \Omega$ as $\rho \to 0$.

LEMMA 2. *Let u and Lu be in L^1_{loc}. Then*

$$u^\rho \to u, \quad Lu^\rho \to Lu \ \text{in} \ L^1_{loc} \ \text{as} \ \rho \to 0.$$

PROOF. The fact that u^ρ is defined only on Ω_ρ does not matter, since the lemma is local and $\Omega_\rho \to \Omega$.

It is well known that $u^\rho \to u$ in L^1_{loc}. Since $(Lu)^\rho \to Lu$, similarly it suffices to show that

$$(LJ^\rho - J^\rho L)u \to 0 \ \text{in} \ L^1_{loc}. \tag{5.1}$$

We shall prove this for each term composing L.

For the term $\partial_j a_{jk} \partial_k$, we have, writing a for a_{jk},

$$\partial_j a \partial_k J^\rho u - J^\rho \partial_j a \partial_k u = \partial_j (aJ^\rho - J^\rho a)\partial_k u, \tag{5.2}$$

since J^ρ and ∂_k commute. Since $\partial_k u \in L^1_{loc}$ by Lemma 1 and $a \in C^1$, (5.2) tends to 0 in L^1_{loc} by a well-known result (see [4]). For the term $\partial_j a_{jk} b_k$ we have, writing $a_{jk} b_k = c$,

$$\partial_j c J^\rho u - J^\rho \partial_j c u = \partial_j (cJ^\rho - J^\rho c)u \to 0 \ \text{in} \ L^1_{loc}$$

in the same way, since $u \in L^1_{loc}$ and $c \in C^1$. Other terms can be dealt with in a similar fashion.

LEMMA 3. *Let u and Lu be in L^1_{loc}. Let $\varepsilon > 0$ and*

$$u_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{1}{2}} \geqq \varepsilon. \tag{5.3}$$

Then

$$L_0 u_\varepsilon \geqq \operatorname{Re}[(\bar{u}/u_\varepsilon)Lu]. \tag{5.4}$$

PROOF. Since $u_\varepsilon \in L^1_{loc}$ and $|\bar{u}/u_\varepsilon| \leqq 1$, (5.4) makes sense in the same way as (1.7). The proof of (5.4) is given in two steps.

I. First assume that u is smooth; $u \in C^2$ is sufficient. Differentiating $u_\varepsilon^2 = u\bar{u} + \varepsilon^2$, we obtain

$$2u_\varepsilon \partial_k u_\varepsilon = (\partial_k u)\bar{u} + u(\partial_k \bar{u}) = (D_k u)\bar{u} + u\overline{(D_k u)}, \tag{5.5}$$

where $D_k = \partial_k - ib_k$ as before. It follows that

$$u_\varepsilon^2 \sum_{j,k}^{1,m} a_{jk}(x)(\partial_j u_\varepsilon)(\partial_k u_\varepsilon) \leqq |u|^2 \sum_{j,k}^{1,m} a_{jk}(x)\,(D_j u)\overline{(D_k u)} \tag{5.6}$$

for each x; the proof is elementary and involves only linear algebra for fixed x.

Next we multiply (5.5) by a_{jk} and differentiate in x_j, obtaining

$$
\begin{aligned}
2u_\varepsilon \partial_j a_{jk} \partial_k u_\varepsilon &+ 2a_{jk}(\partial_j u_\varepsilon)(\partial_k u_\varepsilon) \\
&= \bar{u}\partial_j a_{jk} D_k u + a_{jk}(\partial_j \bar{u})(D_k u) + u\partial_j a_{jk}\overline{D_k u} + a_{jk}(\partial_j u)(\overline{D_k u}) \\
&= \bar{u}(D_j + ib_j)a_{jk}D_k u + a_{jk}(\overline{D_j u} - ib_j\bar{u})(D_k u) \\
&\qquad + u(\bar{D}_j - ib_j)a_{jk}\overline{D_k u} + a_{jk}(D_j u + ib_j u)(\overline{D_k u}).
\end{aligned}
$$

Here all four terms involving b_j explicitly cancel one another. On summing over $j,\ k = 1,\cdots,m$ and noting $a_{jk} = a_{kj}$, we obtain

$$
\begin{aligned}
2u_\varepsilon L_0 u_\varepsilon &+ 2\ \sum a_{jk}(\partial_j u_\varepsilon)(\partial_k u_\varepsilon) \\
&= \bar{u}Lu + u\overline{Lu} + 2\ \sum a_{jk}(D_j u)(\overline{D_k u}).
\end{aligned}
$$

But the second term on the left is not larger than the last term on the right, by virtue of (5.6) and $u_\varepsilon \geqq |u|$. Hence $u_\varepsilon L_0 u_\varepsilon \geqq \operatorname{Re}(\bar{u}Lu)$, and we obtain (5.3) on dividing by $u_\varepsilon > 0$.

II. In the general case, we mollify u to $u^\rho = J^\rho u$. We have $u^\rho \to u$ and $Lu^\rho \to Lu$ in L^1_{loc} as $\rho \to 0$, by Lemma 2. Furthermore, we shall let $\rho \to 0$ along a suitable sequence so that $u^\rho \to u$ a.e. pointwise.

Define $u^\rho_\varepsilon = (u^\rho)_\varepsilon$ by (5.3). A simple calculation shows that $|u^\rho_\varepsilon - u_\varepsilon| \leqq \big||u^\rho| - |u|\big| \leqq |u^\rho - u|$. Hence $u^\rho_\varepsilon \to u_\varepsilon$ as $\rho \to 0$, in L^1_{loc} as well as a.e. pointwise. It follows that

$$
L_0 u^\rho_\varepsilon \to L_0 u_\varepsilon \text{ in } \mathscr{D}'(\Omega). \tag{5.7}
$$

Also

$$
\begin{aligned}
(\bar{u}^\rho/u^\rho_\varepsilon)Lu^\rho - (\bar{u}/u_\varepsilon)Lu &= (\bar{u}^\rho/u^\rho_\varepsilon)(Lu^\rho - Lu) \\
&\quad + [(\bar{u}^\rho/u^\rho_\varepsilon) - (\bar{u}/u_\varepsilon)]Lu \to 0 \text{ in } L^1_{loc},
\end{aligned} \tag{5.8}
$$

because $|\bar{u}^\rho/u^\rho_\varepsilon| \leqq 1$ and $\bar{u}^\rho/u^\rho_\varepsilon \to \bar{u}/u_\varepsilon$ a.e. (use the dominated convergence theorem for the second term on the right of (5.8)). Since (5.4) is true for u replaced by u^ρ, we see from (5.7) and (5.8) that it is true for u; note that the limit of nonnegative distributions is nonnegative.

Proof of Lemma A. The inequality (1.7) follows from (5.4) in the limit as $\varepsilon \to 0$. Indeed, we have $u_\varepsilon \to |u|$ uniformly, so that $L_0 u_\varepsilon \to L_0|u|$ in $\mathscr{D}'$. Also $\bar{u}/u_\varepsilon \to \operatorname{sign}\bar{u}$ boundedly, so that the right hand side of (5.4) tends to $\operatorname{Re}[(\operatorname{sign}\bar{u})Lu]$ in L^1_{loc} (again use the dominated convergence theorem). This gives (1.7) as required.

REFERENCES

1. M. Schechter, *Spectra of Partial Differential Operators*, North Holland Publishers, 1971.

2. B. Simon, *Essential self-adjointness of Schrödinger operators with positive potentials*, Math. Ann., to appear.

3. T. Ikebe and T. Kato, *Uniqueneness of the s lf-adjoint extension of singular elliptic differential operators*, Arch. Rational Mech. Anal. **9** (1962), 77–92.

4. K. O. Friedrichs, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc. **55** (1944), 132–151.

UNIVERSITY OF CALIFORNIA, BERKELEY